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Analysis of the motion of a non-holonomic mechanical system☆

D.N. Zeković

Beograd, Serbia

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ABSTRACT

The motion of a plane non-holonomic mechanical system, consisting of two point masses, which move in such a way that their velocities are mutually perpendicular, is analysed [Zeković D. Examples of non-linear non-holonomic constraints in classical mechanics. Vestnik MGU. Ser. 1. Matematika Mekhanika, 1991; 1:100–3]. The equations of the constraints of such a system are derived, the reactions of the constraints are calculated and the cyclical first integrals are written.

1. Analysis of the constraint equations

The practical realization of a system consisting of two point masses, which move in such a way that their velocities are mutually perpendicular, can be achieved using two knife edges and a weightless rigid construction (“a fork”), as shown in the Fig. 1. This system is non-holonomic[1].

Suppose x_1, y_1, x_2 and y_2 are generalized coordinates. The condition for the velocities of the point masses to be orthogonal can be written in the form

$$\dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2 = 0 \tag{1.1}$$

Two other constraint equations express the conditions for the velocities of the points M_1 and M_2 and the planes of the knife edges to be codirectional:

$$\dot{x}_1\cos\varphi + \dot{y}_1\sin\varphi = 0, \qquad \dot{x}_1\sin\varphi - \dot{y}_2\cos\varphi = 0 \tag{1.2}$$

Constraint equations (1.1) and (1.2) are dependent, and Eq. (1.1) and any of Eqs. (1.2) or both of Eqs. (1.2) can be used. Since the reactions of the constraints $\mathbf{R}_1$ and $\mathbf{R}_2$ are perpendicular to the planes of the knife edges, their scalar product is equal to zero.

The left-hand side of Eq. (1.1) is a homogeneous function of Φ of power two, and this relation can be written in the form

$$\frac{\partial\Phi}{\partial\dot{q}^i}\dot{q}^i = 0$$

where q^i are generalized coordinates, and the possible displacements permitted by this non-linear constraint, we will define, like Chetayev, in the form

$$\frac{\partial\Phi}{\partial\dot{q}^i}\delta\dot{q}^i = 0$$

Here and henceforth summation is carried out over repeated indices, where

$$i, j, k = 1, \ldots, n; \quad \alpha, \beta, \gamma = 1, \ldots, m; \quad \nu, \rho = m+1, \ldots, m+l = n$$

Note that the coefficients of δq^i depend on the velocities.

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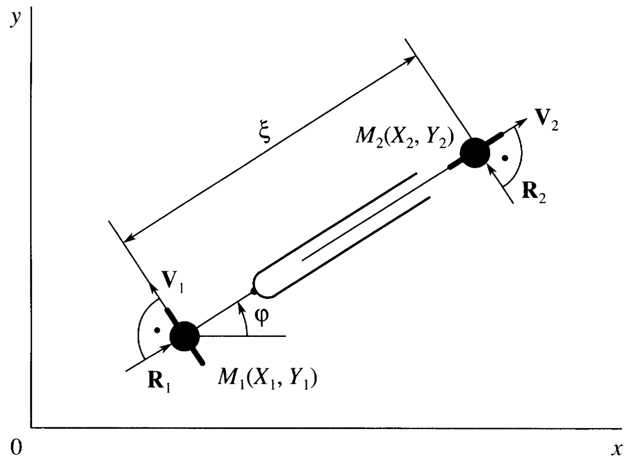

Fig. 1.

2. The trajectories of points of the system

We will write the equations of motion of the system, choosing as the generalised coordinates x_i, y_i, ξ and φ (see the Fig. 1),

$$\dot{x}_1\cos\varphi + \dot{y}_1\sin\varphi = 0, \quad \dot{x}_1\sin\varphi - \dot{y}_1\cos\varphi - \xi\dot{\varphi} = \sqrt{\dot{x}_1^2 + \dot{y}_1^2} - \xi\dot{\varphi} = 0 \tag{2.1}$$

Taking $\dot{\xi}$ and $\dot{\varphi}$ as independent velocities, we obtain

$$\dot{x}_1 = \xi\dot{\varphi}\sin\varphi, \quad \dot{y}_1 = -\xi\dot{\varphi}\cos\varphi \tag{2.2}$$

The kinetic energy of the system, provided that the masses of the points M_1 and M_2 are equal to unity, can be written in the form

$$T = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}(\dot{x}_2^2 + \dot{y}_2^2), \quad T^* = \frac{1}{2}(\xi\dot{\varphi})^2 + \frac{1}{2}(\dot{\xi})^2 \tag{2.3}$$

where T^* is the result of eliminating the dependent velocities in the expression for T using the constraint equations.

The system is a Chaplygin system in its inertial motion, and its equations of motion in Chaplygin form are[2,3]

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}^\alpha} - \frac{\partial T^*}{\partial q^\alpha} + \frac{\partial T^*}{\partial \dot{q}^\nu}\gamma_\alpha^\nu = 0$$

$$\gamma_\alpha^\nu = \frac{\partial \psi^\nu}{\partial q^\alpha} - \frac{d}{dt}\frac{\partial \psi^\nu}{\partial \dot{q}^\alpha}, \quad \dot{q}^\nu = \psi^\nu(q^\alpha, \dot{q}^\alpha), \tag{2.4}$$

$$(\dot{q}^3 = \dot{x}_1 = \xi\dot{\varphi}\sin\varphi = \psi^3, \quad \dot{q}^4 = \dot{y}_1 = -\xi\dot{\varphi}\cos\varphi = \psi^4)$$

Using (2.1)–(2.4), we obtain the following equations of motion

$$\ddot{\xi} = 0, \quad (\xi\dot{\varphi})^{\cdot} = 0$$

These equations can be integrated in succession giving

$$\dot{\xi} = C_1 = \dot{\xi}_0, \quad \xi\dot{\varphi} = C_2 = \xi_0\dot{\varphi}_0 \tag{2.5}$$

$$\xi = \dot{\xi}_0 t + \xi_0, \quad \varphi = \frac{\xi_0\dot{\varphi}_0}{\dot{\xi}_0}\ln(\dot{\xi}_0 t + \xi_0) + C_3 \tag{2.6}$$

In the special case when $\dot{\xi}_0 = 0$, the point M_2 will be at rest at the centre of a circle of radius ξ_0, while the point M_1 will move uniformly over this circle. On the other hand, if the point M_1 is at rest, the point M_2 will move uniformly along a fixed straight line, connecting these points.

We will put $\xi_0 = 1$, $\dot{\xi}_0 = 1$, $\varphi_0 = 0$, $\dot{\varphi}_0 = 1$

From Eqs. (2.2) and (2.6) we obtain the laws of motion of the points M_1 ($s = 1$, $\delta_{21} = 0$)7 and M_2 ($s = 2$, $\delta_{22} = 1$)

$$x_s = \frac{e^{t+1}}{2}\{\sin[\ln(t+1)] - \cos[\ln(t+1)]\} + \delta_{2s}(t+1)\cos[\ln(t+1)],$$

$$y_s = -\frac{e^{t+1}}{2}\{\sin[\ln(t+1)] + \cos[\ln(t+1)]\} + \delta_{2s}(t+1)\sin[\ln(t+1)]$$

3. Integrals that are linear in the momenta (cyclic coordinates)

We will show that, in the system considered, for each of the two linear integrals (2.5), we can choose generalized coordinates such that one of them will be cyclical (in the sense given earlier[4]), and the integral corresponding to it will be cyclical.

We will choose as the generalized coordinates of the system

$$x_1 = q^1, \quad y_1 = q^2, \quad x_2 = q^3, \quad y_2 = q^4$$

and we will write the constraint equations in the form

$$f^1 = \dot{q}^1\cos\varphi + \dot{q}^2\sin\varphi = 0, \quad f^2 = \dot{q}^1\dot{q}^3 + \dot{q}^2\dot{q}^4 = 0$$

The equations of motion allow of two first integrals that are linear and homogeneous in the velocities

$$(q^3 - q^1)(\dot{q}^4 - \dot{q}^2) + (q^4 - q^2)(\dot{q}^3 - \dot{q}^1) = \text{const},$$

$$\frac{(q^3 - q^1)(\dot{q}^3 - \dot{q}^1) + (q^4 - q^2)(\dot{q}^4 - \dot{q}^2)}{\sqrt{(q^3 - q^1)^2 + (q^4 - q^2)^2}} = \text{const}$$

The Lagrange and Hamilton functions have the form

$$L = \frac{1}{2}[(\dot{q}^1)^2 + (\dot{q}^2)^2] + \frac{1}{2}[(\dot{q}^3)^2 + (\dot{q}^4)^2], \quad H = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2),$$

We will choose as the new coordinates Q^1 and Q^2 the Cartesian coordinates of the point M_2, as Q^3 we will choose the length of the section M_1M_2 and we will choose as Q^4 the angle of inclination of the section M_1M_2 to the x axis. The corresponding conversion formulae have the form

$$q_1 = Q^1 - Q^3\cos Q^4, \quad q_2 = Q^2 - Q_3\sin Q^4, \quad q_3 = Q^1, \quad q_4 = Q^2$$

The Lagrange and Hamilton functions, written, taking into account the constraint equations,

$$F^3 = \dot{Q}^1\cos Q^4 + \dot{Q}^2\sin Q^4 - \dot{Q}^3 = 0,$$

$$F^4 = \dot{Q}^1\sin Q^4 - \dot{Q}^2\cos Q^4 = 0$$

take the following form in the new variables

$$L^* = \frac{1}{2}[(\dot{Q}^1)^2 + (\dot{Q}^2)^2] + \frac{1}{2}(\dot{Q}^3\dot{Q}^4)^2, \quad H^* = \frac{1}{2}P_1^2 + \frac{1}{2}P_2^2 + \frac{1}{2}\frac{1}{(Q^3)^2}P_4^2$$

Since

$$\frac{\partial H^*}{\partial Q^4} = 0, \quad A_4^3 = \frac{\partial f^3}{\partial \dot{q}^i}\frac{\partial q^i}{\partial Q^4} = \frac{\partial F^3}{\partial \dot{Q}^4} = 0, \quad A_4^4 = \frac{\partial f^4}{\partial \dot{q}^i}\frac{\partial q^i}{\partial Q^4} = \frac{\partial F^4}{\partial \dot{Q}^4} = 0,$$

the coordinate Q^4 is cyclical. We have the cyclic integral $P_4 = (Q^3)^2\dot{Q}^4 = \text{const}$, i.e. $\xi^2\dot{\varphi} = \text{const}$.

Choosing as the new coordinates Q^1 and Q^2 the Cartesian coordinates of the point M_1 and the coordinates Q^3 and Q^4, as shown above, we obtain the following conversion formulae

$$q_1 = Q^1, \quad q_2 = Q^2, \quad q_3 = Q^1 + Q^3\cos Q^4, \quad q_4 = Q^2 + Q^3\sin Q^4$$

As before, we conclude that the coordinate Q^3 is cyclical. We have the cyclic integral $P_4 = (\dot{Q}^3) = \text{const}$, i.e., $\dot{\xi} = \text{const}$.

4. The stationarity of the Hamilton action

For the system being considered, we will check that the following conditions[5] are satisfied

$$\lambda_\nu\gamma_\alpha^\nu = 0 \tag{4.1}$$

For these conditions the Hamilton principle will be the principle of stationary action. Here γ_α^ν are terms of non-holonomicity in Eqs. (2.4) and λ_ν are coefficients of the Euler–Lagrange variational problem

$$\delta\int_{t_0}^{t}(L+\lambda_\nu f^\nu)dt = 0;\quad f^\nu = \dot{q}^\nu - \psi^\nu(q^i, \dot{q}^\alpha) = 0$$

The equations of the extremals

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \lambda_\nu\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\right) - \dot{\lambda}_\nu\frac{\partial f^\nu}{\partial \dot{q}^i}$$

in general are not identical with the equations of motion of the system. They can be written in the form[3]

$$\frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}^\alpha} - \frac{\partial L^*}{\partial q^\alpha} - \frac{\partial L^*}{\partial q^\nu}\frac{\partial \psi^\nu}{\partial \dot{q}^\alpha} + \left(\frac{\partial L}{\partial \dot{q}^\nu} + \lambda_\nu\right)\gamma_\alpha^\nu = 0, \tag{4.2}$$

$$\frac{d\lambda_\nu}{dt} + \frac{\partial \psi^p}{\partial q^\nu}\lambda_p + \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\nu} - \frac{\partial L}{\partial q^\nu} = 0, \tag{4.3}$$

where

$$L^* = L^*(q^i, \dot{q}^\alpha) = L_{\dot{q}^\nu = \psi^\nu},\qquad \gamma_\alpha^\nu = \frac{\partial \psi^\nu}{\partial q_\alpha} + \frac{\partial \psi^\nu}{\partial q^p}\frac{\partial \psi^p}{\partial \dot{q}^\alpha} - \frac{d}{dt}\frac{\partial \psi^\nu}{\dot{q}^\alpha}$$

When conditions (4.1) are satisfied, Eqs. (4.2) take the form of the Voronets equations of motion of a non-holonomic system. Putting

$$q^1 = \xi,\quad q^2 = \varphi,\quad q^3 = x_2,\quad q^4 = y_2$$

we can represent the constraint equations in the form

$$\dot{q}^3 = \dot{\xi}\cos\varphi = \psi^3,\quad \dot{q}^4 = \dot{\xi}\sin\varphi = \psi^4$$

We will write the non-holonomicity coefficients (4.4) as follows:

$$\gamma_1^3 = \dot{\varphi}\sin\varphi,\quad \gamma_1^4 = -\dot{\varphi}\cos\varphi,\quad \gamma_2^3 = -\dot{\xi}\sin\varphi,\quad \gamma_2^4 = \dot{\xi}\cos\varphi \tag{4.4}$$

Consequently, conditions (4.1) of the stationarity of the action are

$$\lambda_3\gamma_1^3 + \lambda_4\gamma_1^4 = (\lambda_3\sin\varphi - \lambda_4\cos\varphi)\dot{\varphi} = 0,$$

$$\lambda_3\gamma_2^3 + \lambda_4\gamma_2^4 = (-\lambda_3\sin\varphi + \lambda_4\cos\varphi)\dot{\xi} = 0 \tag{4.5}$$

Dropping the trivial case $\dot{\xi} = \dot{\varphi} = 0$, we reduce conditions (4.5) to the form

$$\lambda_3\sin\varphi - \lambda_4\cos\varphi = 0 \tag{4.6}$$

It follows from Eq. (4.3) that

$$\lambda_3 = C_1 - \dot{x}_2,\quad \lambda_4 = C_2 - \dot{y}_2\quad (C_1, C_2 - \text{are constants})$$

and hence limitation (4.6) (taking the constraints into account) reduces to the equation

$$C_1\sin\varphi - C_2\cos\varphi = 0$$

i.e., φ = const. So, the particular motion in which the point M_1 is at rest while the point M_2 moves along a straight line, yields a stationary value of the Hamilton action.If we take as the generalized coordinates

$$q^1 = \xi,\quad q^2 = \varphi,\quad q^3 = x_1,\quad q^4 = y_1$$

the constraint equations can be written as

$$\dot{q}^3 = \xi\dot{\varphi}\sin\varphi = \psi^3,\quad \dot{q}^4 = -\xi\dot{\varphi}\cos\varphi = \psi^4 \tag{4.7}$$

where the terms of non-holonomicity retain the form (4.4), and, consequently, the stationarity conditions (4.5) do not change. We obtain from Eqs. (4.3)

$$\lambda_3 = C_0 - \dot{x}_1,\quad \lambda_4 = C_1 - \dot{y}_1,$$

as a consequence of which condition (4.6) becomes the equation

$$\lambda_3 \sin\varphi - \lambda_4 \cos\varphi = C_0 \sin\varphi - C_1 \cos\varphi - \xi\dot{\varphi} = C_0 \sin\varphi - C_1 \cos\varphi - C_2 = 0$$

(the second integral of (2.5) has been used). It is only satisfied for the already known case $\varphi = \text{const}$.

However, there is one more case when, in the particular motion of the system, the Hamilton action takes a stationary value. This case is $\dot{\xi} = 0$: the point M_2 is at rest while the point M_1 moves in a circle. The point is that in this case the constraints (4.7) are integrable, and the system is holonomic, as a result of which the action stationarity conditions are satisfied. This case is not obtained from the preceding analysis due to the structure of the coefficients γ_α^ν (4.4), because the coefficient ξ occurs in terms the sum of which is equal to zero.

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